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Excitation of resonators by electron beams

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Abstract

In this paper the main consequences of the vector theory of excitation of resonators by particle beams are presented. Some features of excitation of broadband radiation in longitudinal modes of the enclosed and open resonators are discussed.

1 Introduction

The excitation of resonators is described by Maxwell equations in vacuum [1] - [3]

$$div\vec{E} = 4\pi\rho \quad (a) \qquad rot\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \quad (b),$$
$$rot\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \quad (c), \qquad div\vec{H} = 0 \quad (d). \tag{1}$$

These equations are a set of two vector and two scalar equations for vectors of electric $\vec{E}(\vec{r},t)$ and magnetic $\vec{H}(\vec{r},t)$ field strengths or eight equations for six independent components of the electric and magnetic fields. We suppose that the charge density $\rho(\vec{r},t)$ and current density $\vec{J}(\vec{r},t)$ are given values. It means that only four components of the electromagnetic field strengths are independent.

The solution of these equations includes transverse electromagnetic field strengths of free electromagnetic waves $\vec{E}^{tr}$, $\vec{H}^{tr}$ and accompanied longitudinal electric field strengths $\vec{E}^{l}$ of Coulomb fields of the beam crossing the resonator. Transverse electromagnetic field strengths excited by the beam in the resonator comply the condition $div\vec{E}^{tr} = div\vec{H}^{tr} = div\vec{H} = 0$. Longitudinal electric field strength comply the condition $rot\vec{E}^{l} = 0$, $div\vec{E}^{l} = 4\pi\rho$ [2] - [5][1]. Free electromagnetic fields in resonators are solutions of homogeneous Maxwell equations ($\vec{\jmath} = \rho = 0$) with corresponding boundary conditions. These solutions are a sum of eigenmodes of the resonator which include a discrete set of eigenfrequences ω_λ and corresponding to them functions $\vec{E}_\lambda(\vec{r},t), \vec{H}_\lambda(\vec{r},t)$ for the electric and magnetic field strengths (further we will omit the superscripts tr and l in the fields). The subscript λ includes three numbers (m,n,q) corresponding to transverse and longitudinal directions of the resonator axis. In the case of open resonators the transverse electromagnetic TEM_{mnq} modes are excited. When the number q is very high then

[1]In general case transverse fields are not only free electromagnetic waves. Both a static magnetic field, a magnetic field accompanying a homogeneously moving particle and arbitrary time depended magnetic field are transverse one. A part of the Coulomb electrical field accompanying a relativistic particle is transverse one. The most simple example of the transverse electric field strength is the electric field strength of the homogeneously moving relativistic particle $\vec{E}^{tr} = \vec{E} - \vec{E}^{l}$, where $\vec{E} = e\vec{r}/\gamma^2 r^{*3}$, $\vec{E}^{l} = e\vec{r}/r^3$, $\vec{r}$ is the radius vector directed from the particle to the observation point, $\gamma = \sqrt{1-\beta^2}$ relativistic factor of the particle, $R^* = (x-vt)^2 + (1-\beta^2)(x^2+y^2)$, $\beta = v/c$, v the velocity of the particle [1], [2]. After a particle beam cross a resonator then only transverse free electromagnetic waves stay at the resonator.

this number is omitted. Usually in the open resonators many longitudinal modes are excited even in the case of free-electron lasers emitting rather monochromatic radiation.

The solution of the problem of excitation of resonators is simplified by introduction of a transverse vector potential $\vec{A}(\vec{r},t) = \sum_\lambda \vec{A}_\lambda(\vec{r},t)$ of free electromagnetic fields in Coulomb gauge $div\vec{A} = 0$, where scalar potential $\varphi = 0$ when $\rho = 0$ (here we omitted the superscripts tr and l in the vectors $\vec{A}^{tr}$). The corresponding wave equation for this vector can be solved by the method of separation of variables when we suppose $\vec{A}_\lambda(\vec{r},t) = q_\lambda(t)\cdot\vec{A}_\lambda(\vec{r})$, where $q_\lambda(t)$ is the amplitude of the vector potential and $\vec{A}_\lambda(\vec{r})$ is the eigenfunction of the resonator normalized by the condition $\int |\vec{A}_\lambda(\vec{r})|^2 dV = 1$. In this case the total free electromagnetic field in the resonator is described by the expression $\vec{A}(\vec{r},t) = \sum_\lambda \vec{q}_\lambda(t)\vec{A}_\lambda(\vec{r})$.

The electric and magnetic field strengths of the transverse free fields in resonators can be expressed through the vector potential in the form $\vec{E}_\lambda(\vec{r},t) = -d\vec{A}_\lambda(\vec{r},t)/d\,ct = -\dot{q}_\lambda(t)\cdot\vec{A}_\lambda(\vec{r})/c$, $\vec{H}_\lambda(\vec{r},t) = rot\vec{A}_\lambda(\vec{r},t) = q_\lambda(t)\cdot rot\vec{A}_\lambda(\vec{r})$, where $\dot{q}_\lambda(t) = d\,q_\lambda(t)/dt$. When the charge and current densities are in the resonator then a scalar φ_σ and a longitudinal vector potential $\vec{A}^l$ ($rot\vec{A}^l = 0$) determine Coulomb fields of the beam in the resonator. We are not interesting them in this paper.

When active and diffractive losses in the open resonator are absent then the vector potential of a free electromagnetic field in the resonator excited by the beam can be presented in the form

$$\vec{A}(\vec{r},t) = \sum_\lambda q_{m\lambda}\vec{A}_\lambda(\vec{r})e^{i\omega_\lambda t}, \tag{2}$$

where the coefficient $q_{m\lambda}$ is the amplitude of the excited eigenmode.

The electromagnetic fields excited by the electromagnetic beam are determined by the non-homogeneous Maxwell equations or the corresponding equation for the vector potential

$$\Delta\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\frac{4\pi}{c}\vec{J}(\vec{r},t). \tag{3}$$

The solution of the Eq(3) can be found in the form $\vec{A}(\vec{r},t) = \sum_\lambda q_\lambda(t)\vec{A}^{tr}_\lambda(\vec{r}) + \sum_\sigma q_\sigma(t)\vec{A}^l_\sigma(\vec{r})$ (here we stayed superscripts tr and l and used the conditions $\vec{A}_{\lambda,\sigma}(\vec{r},t) = q_{\lambda,\sigma}(t)\cdot\vec{A}_{\lambda,\sigma}(\vec{r})$). If we will substitute this expression into equation (3), integrate over the volume of the resonator, use the condition of normalization $\int |\vec{A}_\lambda(\vec{r})|^2 dV = \int |\vec{A}_\sigma(\vec{r})|^2 dV = 1$, $\int \vec{A}_\lambda(\vec{r})\vec{A}_{\lambda'}(\vec{r})dV = \int \vec{A}_\lambda(\vec{r})\vec{A}_\sigma(\vec{r})dV = \int \vec{A}_\sigma(\vec{r})\vec{A}_{\sigma'}(\vec{r})dV = 0$ and take into account that the vector $\vec{A}_\lambda$ comply with the condition $\Delta\vec{A}_\lambda = -(\omega_\lambda/c)^2\vec{A}_\lambda$ then we will receive the equation for change of the amplitude of the eigenmode q_λ for free fields in the resonator

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{4\pi}{c}\int_V \vec{J}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV. \tag{4}$$

The expression (4) is the equation of the oscillator of unit mass excited by a force $f(t) = (4\pi/c)\int_V \vec{J}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV$. It describes the excitation of both enclosed and open resonators [3] - [5]. The same expression for force determine the excitation of waveguides [2].

The eigenmodes of the rectangular resonators (cavities) were discovered by J.Jeans in 1905 when he studied the low of thermal emission. The equations (4) was used later for quantization of the electromagnetic field in quantum electrodynamics [3].

2 Emission of electromagnetic radiation by electron beams in open resonators

The equation (4) does not take into account the energy losses of the emitted radiation in the resonator. These losses can be introduced through the quality of the resonator Q_λ

$$\ddot{q}_\lambda + \frac{\omega_r}{Q_\lambda}\dot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{4\pi}{c}\int_V \vec{J}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV, \tag{5}$$

where in the case of the open resonator $\omega_r = 2\pi/T$, $T = 2L/c$ is the period of oscillations of the light wavepacket between the resonator mirrors when it passes along the axis of the resonator (notice that in general case the frequencies $\omega_\lambda = \omega_{mnq}$ depend on m, n, q and slightly differ from frequencies $\omega_r q$). Here we have introduced a version of a definition of a resonator quality connected with the frequency ω_r. Another version of a quality is usually connected with the frequency ω_λ. Our definition is more convenient for the case of free-electron lasers using open resonators.

Using (5) we can derive the expression for the energy balance in the resonator. For this purpose we can multiply this equation by $\dot{q}_\lambda$ and integrate over the volume of the resonator. Then we receive the equation

$$\frac{1}{2}\frac{d}{dt}[\dot{q}_\lambda^2 + \omega_\lambda^2 q_\lambda^2] + (\frac{\omega_r}{Q_\lambda})^2\dot{q}_\lambda^2 = 4\pi\int_V \vec{J}(\vec{r},t)\vec{E}_\lambda(\vec{r},t)dV. \tag{6}$$

If we take into account that $\vec{E}_\lambda(\vec{r},t) = -\dot{q}_\lambda(t)\cdot\vec{A}_\lambda(\vec{r})/c$, $\vec{H}_\lambda(\vec{r},t) = rot\vec{A}_\lambda(\vec{r})$, $rot\vec{A}_\lambda = \omega_\lambda\vec{A}_\lambda/c$, $\int|\vec{A}_\lambda(\vec{r})|^2dV = 1$ then the energy of the free electromagnetic field in the resonator can be presented in the form $\varepsilon_\lambda^{em} = \int[(|\vec{E}_\lambda|^2 + |\vec{H}_\lambda|^2)/8\pi]\; dV = [\dot{q}_\lambda^2 + \omega_\lambda^2 q_\lambda^2]/8\pi c^2$ and the equation (7) can be presented in the another form

$$\dot{\varepsilon}_\lambda^{em} + (\omega_r/Q_\lambda)\varepsilon_\lambda^{em} = \int_V \vec{J}(\vec{r},t)\vec{E}_\lambda(\vec{r},t)dV. \tag{7}$$

The equation (5) is the pendulum equation with a friction. It determine the time evolution of the electromagnetic field stored at the resonator, when the time dependence of the beam current $\vec{J}(\vec{r},t)$ is given. The amplitude $q_\lambda(\vec{r},t)$ according to (5) is determined by the coefficient of expansion of the given current into series of eigenfunctions of the resonator. Notice that the value $\vec{A}_\lambda[\vec{r}_e(t)]$ depends on t only through $\vec{r}_e(t)$ and the value $\vec{E}_\lambda[t,\vec{r}_e(t)] = -\dot{q}_\lambda(t)\cdot\vec{A}_\lambda[\vec{r}_e(t)]/c$ depends on t directly through $q_\lambda(t)$ and through $\vec{r}_e(t)$.

In the case of one particle of a charge "e" the beam current density $\vec{J}(\vec{r},t) = e\vec{v}(t)\delta[\vec{r}-\vec{r}_e(t)]$. In this case the force $f(t) = e\vec{v}[\vec{r}_e(t)]\vec{A}_\lambda[\vec{r}_e(t)]$ and the power transferred from the electron beam to the resonator wave mode λ excited in the resonator $P_\lambda(t) = \int_V \vec{J}(\vec{r},t)\vec{E}_\lambda(\vec{r},t)dV = e\sum_i \vec{v}_{e\,i}(t)\vec{E}_\lambda[(\vec{r}_{e,1}(t),t)]$. Using these expressions of force and power for all electrons "i" of the beam we can present the equations (5), (7) in the form

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = \frac{4\pi e}{c}\sum_i \vec{v}_{e\,i}(t)\vec{A}_\lambda[\vec{r}_{e\,i}(t)], \tag{8}$$

$$\dot{\varepsilon}_\lambda^{em} + (\omega_r/Q_\lambda)\varepsilon_\lambda^{em} = e\sum_i \vec{v}_{e\,i}(t)\vec{E}_\lambda[(\vec{r}_{e\,i}(t),t)]. \tag{9}$$

It follows from (5), (7) and (8), (9) that transverse resonator modes are excited only in the case when the force $f(t) \neq 0$ and the power $P_\lambda(t) \neq 0$ that is when the particle trajectory passes through the regions where the corresponding resonator modes have large intensities and when the particle velocity has transverse and/or longitudinal components directed along the direction of the electric field strength. Open resonators on the level with enclosed ones have modes with longitudinal components of electric field strength (see Appendix). It means that open resonators can be excited even in the case when the particle trajectory have no transverse components and its velocity is directed along the axis of the resonator[2]. Using external fields of a single bending magnet can increase the power of the generated radiation. Both in the case of lack of a banding magnet and presence of one bending magnet the broadband radiation is emitted. The experiment confirms this observations [7]. Using external fields of undulators and beams bunched at frequencies of the emitted radiation can lead to emission of rather monochromatic radiation.

In the simplest case when the beam current density $\vec{J}(\vec{r},t)$ is a periodic function of time then the force can be expanded in the series $f(t) = \int_V \vec{J}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV = \sum_{\nu=-\infty}^{\infty} f_{\lambda\nu}\exp[i(\nu\omega_b t - \varphi_{\lambda\nu})]$, where $\omega_b = 2\pi/T_b$ and T_b are a period and frequency of the current density oscillation accordingly, $f_{\lambda\nu} = (1/T_b)\int_{-T_b/2}^{T_b/2} f(t)\exp(i\nu\omega_b t)dt$, are the known coefficients, $\varphi_{\lambda\nu}$ phase. The value $f_{\lambda\,-\nu} = f^*_{\lambda\,\nu}$, where $f^*_{\lambda\nu}$ is the complex conjugate of $f_{\lambda\nu}$. The solution of the equation (5) for the case of the established oscillations ($t \gg Q_\lambda T_b$) is

$$q_\lambda(t) = \sum_{\nu=1}^{\infty} A_{\lambda\nu}\exp[i(\nu\omega_b t - \theta_{\lambda\nu})], \tag{10}$$

where

$$A_{\lambda\nu} = \frac{f_{\lambda\nu}}{\sqrt{(\omega_\lambda^2 - \nu^2\omega_b^2)^2 + (\nu\omega_r\omega_b/Q_\lambda)^2}},$$

$$\theta_{\lambda\nu} = \varphi_{\lambda\nu} + arctg\frac{\nu\omega_r\omega_b}{Q_\lambda(\omega_\lambda^2 - \nu^2\omega_b^2)}.$$

It follows from the equation (10) that the maximum of the amplitude of the vector potential $A_{\lambda\nu} = Q_\lambda f_{\lambda\nu}/\omega_r\omega_\lambda^2$ takes place at resonance $\nu\omega_b = \omega_\lambda = \omega_{mnq} \simeq \omega_r q$. Notice that all modes λ are excited at the same frequency ω_b of the oscillator. In general case $\omega_b \neq \omega_\lambda$.

The equation (10) is the first order linear equation of the energy change in the resonator excited by the electron beam. It follows from this equation that after switching off the beam current at some moment t_0 ($\vec{J}(\vec{r},t)|_{t>t_0} = 0$) the energy in the resonator will be changed by the law $\varepsilon^{em}_\lambda = \varepsilon^{em}_{\lambda,/0}\exp[-(t-t_0)/\tau]$, where $\tau = Q_\lambda/\omega_r$, $\varepsilon^{em}_{\lambda 0} = \varepsilon^{em}_\lambda|_{t=t_0}$. On the contrary after switching on the beam current at some moment t_0 the energy in the resonator will be changed by the law $\varepsilon^{em}_\lambda = \varepsilon^{em}_{\lambda m}(1-\exp[-(t-t_0))/\tau]$, where the energy of the electromagnetic field in the resonator $\varepsilon^{em}_{\lambda m}$ is determined by the parameters of the resonator and the beam.

The considered example describes the emission of an oscillator or a system of oscillators which are in phase with the excited mode and have zero average velocity (trajectory has a form $r_e = r_{e0} + \vec{\imath}a_0\cos\omega_0 t$, where $\vec{\imath}$ is the unite vector directed along the axis x). More complicated examples of trajectories of particles using for excitation of resonators by electron beams can be considered (the arc of circle, sine- or helical-like trajectories in bending magnets and undulators).

3 Vector TEM modes of open resonators

The theory of high quality open resonators does not differ from enclosed ones. But eigenmodes of open resonators have some unique features. The spectrum of the open resonators is rarefied, the operating mode spectrum has maximum selectivity. The dimensions of open resonators are much higher then the excited wavelengths and the dimensions of the enclosed resonators are of the order of excited wavelengths. The quality of open resonators at the same wavelengths is higher then enclosed ones.

[2] In this case the transition radiation is emitted by particles when they pass the walls of a resonator. The electromagnetic radiation will be emitted in the form of thin spherical layers at the first and second resonator mirrors [6]. It will be reflected then repeatedly by resonator mirrors. The expansion of the electromagnetic fields of the spherical layers will be described by the series (2).

There are some methods of calculation of TEM modes in open resonators. Usually scalar wave equations are investigated [8], [9]. There is a small information in technical publications about distribution of vectors of the electric and magnetic field strengths in such resonators. In this section we search some distributions. In the Appendix the foundations of the excitation of resonators by electron beams are presented .

We will present the result for the Cartesian coordinates. In this case the solution of the scalar wave equation (24) (see Appendix) has a form [10]

$$V_{mn}(x,y,z) = \frac{C}{\sqrt{w_x(z)w_y(z)}} H_m\left(\frac{\sqrt{2}x}{w_x(z)}\right) H_n\left(\frac{\sqrt{2}y}{w_y(z)}\right) \cdot$$

$$\exp\left\{\frac{ik}{2}\left(\frac{x^2}{q_x(z)} + \frac{y^2}{q_y(z)}\right) - i(m+\frac{1}{2})arctg\frac{\lambda z}{\pi w_{0x}^2} - i(n+\frac{1}{2})arctg\frac{\lambda z}{\pi w_{0y}^2}\right\} \tag{11}$$

and for the cylindrical coordinates

$$V(r,\phi,z) = C\left(\frac{r}{w(z)}\right)^m \begin{pmatrix}\sin m\phi\\ \cos m\phi\end{pmatrix} L_n^m\left(\frac{2r^2}{w^2(z)}\right) \exp\left\{\frac{ikr^2}{q(z)} - i(m+2n+1)arctg\frac{\lambda z}{\pi w_0^2}\right\} w(z)^{-1}, \tag{12}$$

where H_m, H_n are the Hermittian polynomials, L_n^m the Lagerian polynomials, $\lambda = 2\pi c/\omega$ is the wavelength, C = constant,

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2(z)}, \qquad R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right], \qquad w^2(z) = w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right].$$

In (11), (12) $R(z)$ is the radius of the wave front of Gaussian beam, $w(z)$ the radius of the beam, $w_0(z)$ the radius of the waist of the beam.

At $m = n = 0$ we have the main mode of the Gaussian beam. If $w_{0x} = w_{0y} = w_0$ then the main modes for the Cartesian and cylindrical coordinates are the same

$$U(x,y,z) = \frac{C}{w(z)} \exp\left\{-\frac{x^2+y^2}{w^2(z)} + \frac{ik}{2}\frac{x^2+y^2}{R(z)} - i\,arctg\frac{\lambda z}{\pi w_0^2}\right\} \exp^{i(kz-\omega t)}. \tag{13}$$

We have the solutions (11), (12) of the scalar wave equation (24) for the space limited beam. Now we can find vectors of the electric and magnetic field strengths using the expressions (23) and possible ways of construction of Hertz vectors. Let us suppose the next compositions with the electric Hertz vector assuming that magnetic Hertz vector is zero:

1) $\Pi_x^e = U(x,y,z)$, $\Pi_y^e = \Pi_z^e = 0$.

2) $\Pi_x^e = 0$, $\Pi_y^e = U(x,y,z)$, $\Pi_z^e = 0$.

3) $\Pi_x^e = 0$, $\Pi_y^e = 0$, $\Pi_z^e = U(x,y,z)$.

In the first case

$div\vec{\Pi} = \partial\Pi_x/\partial x = \partial V/\partial x \exp[i(kz-\omega t)]$, $(rot\vec{\Pi})_x = 0$,
$(rot\vec{\Pi})_y = (\partial V/\partial z + ikV)\exp[i(kz-\omega t)]$, $(rot\vec{\Pi})_z = -(\partial V/\partial y)\exp[i(kz-\omega t)]$

and

$$E_x^1 = \partial^2 V/\partial x^2 + k^2 V,\ E_y^1 = \partial^2 V/\partial x \partial y,$$
$$E_z^1 = \partial^2 V/\partial x \partial z + ik\partial V/\partial x,\ H_x^1 = 0,\ H_y^1 = ik\partial V/\partial z - k^2 V,\ H_z^1 = ik\partial V/\partial y.$$

The upper superscript shows the first composition of the Hertz vector. A common multiple $\exp[i(kz - \omega t)]$ for all field components is omitted.

The values $\partial^2 V/\partial x_i \partial x_k \ll k\partial V/\partial x_i \ll k^2 V$. That is why in this case $E_x^1 \gg E_y^1, E_z^1$, $H_y^1 \gg H_z^1$.

The second case does not differ from the first one. It is necessary to substitute variable x by y and vise versa.

In the third case

$$div\vec{\Pi} = \partial \Pi_x/\partial z = \partial V/\partial x \exp[i(kz - \omega t)],\ (rot\vec{\Pi})_x = \partial V/\partial y \exp[i(kz - \omega t)],$$
$$(rot\vec{\Pi})_y = -\partial V/\partial x \exp[i(kz - \omega t)],\ (rot\vec{\Pi})_z = 0$$

and

$$E_x^3 = \partial^2 V/\partial x \partial z + ik\partial V/\partial x,\ E_y^3 = \partial^2 V/\partial z \partial y + \partial V/\partial y,$$
$$E_z^3 = 2ik\partial V/\partial z,\ H_x^3 = ik\partial V/\partial y,\ H_y^3 = -ik\partial V/\partial x,\ H_z^3 = 0.$$

It follows that in the case of the main mode the electric and magnetic field strengths corresponding to the electric Hertz vector have components:

$$E_x^1 = k^2 U(x,y,z), \quad E_y^1 \simeq 0, \quad E_z^1 = 2ikx\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z),$$

$$H_x^1 \simeq 0, \quad H_y^1 = -k^2 U(x,y,z), \quad H_z^1 = 2iky\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z)$$

$$E_x^2 =\simeq 0, \quad E_y^2 = k^2 U(x,y,z), \quad E_z^2 = 2iky\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z),$$

$$H_x^2 = -k^2 U(x,y,z), \quad H_y^2 \simeq 0, \quad H_z^2 = 2ikx\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z)$$

$$E_x^3 = 2ikx\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z), \quad E_y^3 = -2iky\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z), \tag{14}$$

$$E_z^3 = 2ik\left\{\frac{4\lambda(x^2+y^2)z}{(\pi w_0)^2 w^3(z)} + \frac{ik(x^2+y^2)}{2R^2(z)}\left[1 - \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] - \frac{i\lambda}{\pi w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]}\right\}U(x,y,z),$$

$$H_x^3 = 2iky\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z), \quad H_y^3 = -2iky\left[\frac{1}{w^2(z)} + \frac{ik}{R(z)}\right]U(x,y,z), \quad H_z^3 = 0.$$

The electric and magnetic field strengths received from magnetic Hertz vector can be received from the fields (14) as well. For this purpose we can take the vector of the electric field strength received from magnetic Hertz vector equal to the negative value of the magnetic field strength received from the electric Hertz vector $\vec{E}' \to -\vec{H}$ and by analogy we can take $\vec{H}' \to \vec{E}$.

The general solution for the electromagnetic field strength of the main mode of Gaussian beam TEM_{00} can be presented in the form

$$\vec{E} = c_1\vec{E}^1 + c_2\vec{E}^2 + c_3\vec{E}^3 - c_4\vec{H}^1 - c_5\vec{H}^2 - c_6\vec{H}^3,$$

$$\vec{H} = c_1\vec{H}^1 + c_2\vec{H}^2 + c_3\vec{H}^3 + c_4\vec{E}^1 + c_5\vec{E}^2 + c_6\vec{E}^3, \tag{15}$$

where c_i are the arbitrary coefficients determined by the conditions of excitation of the mode by the electron beam. Waves determined by the only coefficient c_i (when another ones are equal to zero) can be excited independently.

Higher modes in the open resonator will be described by the expressions (15) and by the expressions similar to (14) for the electromagnetic field strengths of the main mode. They will form orthogonal and full set of fundamental waves. The arbitrary wave may be expanded into these waves. Of cause, real electric and magnetic field strengths are determined by the real part of the expression (15).

In the open resonators the same Gaussian beams are excited. They propagate between mirrors both in z and in $-z$ directions. However the resonators will be excited on discrete set of eigenfrequences (wavelengths) [10].

We can see that according to (14) all considered waves $\vec{E}^i, \vec{H}^i$ are transverse. At the same time they have longitudinal components. This is the general property of the convergent and divergent waves [4], [10]. Such waves have longitudinal components which permit the lines of the electric and magnetic field strengths to be closed.

The fields $\vec{E}^1, \vec{H}^1$ describe an electromagnetic wave with one direction of polarization and the fields $\vec{E}^2, \vec{H}^2$ with another one. They have high transverse components of the electric and magnetic field strengths and zero longitudinal components on the axis z.

Electromagnetic fields $\vec{E}^3, \vec{H}^3$ are a new kind of fields. They have zero transverse components of the electric and magnetic field strengths and high value longitudinal component of the electric field strength at the axis z (similar to the wave E_{01} at the axial region of the cylindrical waveguide). It means that in this case the lines of the electric and magnetic field strengths are closed in the directions both at the central part of the beam propagation that is near to the axis z and far from the axis that is near to the region of theirs envelopes (caustics)[3].

Usually the scalar functions $V(x, y, z)$ or $U(x, y, z) = V(x, y, z)\exp[i(kz - \omega t)]$ are used when the modes in open resonators are investigated [4], [8], [9]. It was supposed that the waves are transverse ones and the values of the electromagnetic field strengths are distributed near the same way as the values of the scalar functions. At that some features like the existence of the wave $\vec{E}^3, \vec{H}^3$ were hidden. Such waves have longitudinal components of the electric field strength and hence can be excited through the transition radiation emitted on the inner sides of the resonator walls by an electron homogeneously moving along the axis z. Such excitation was observed in the experiments published in [7].

4 Conclusion

Open resonators permit an effective generation of broadband radiation at the main and/or other transverse modes under conditions when many longitudinal modes are excited. The longitudi-

[3]Notice that usually the divergent waves with high directivity emitted by antennas are described and drawn by the lines of the electric and magnetic field strengths which are closed in the directions far from the axis of the beam propagation near to the region of theirs envelops.

nal modes are limited in the longwavelength region by the diffraction losses and in the short wavelength region by the longitudinal electron beam dimensions (coherence conditions). Open resonators can be excited in the case when the external fields in the resonator are absent and the particle trajectory is directed along the axis of the resonator. Using external fields of a single bending magnet can increase the power of the generated radiation [7].

Appendix

Generation and propagation of electromagnetic waves in vacuum is described by Maxwell equations (1). We noticed above that these equations are a set of eight equations for six independent components of the electric and magnetic fields. Only four components of the electromagnetic field are independent. These equations added with initial and boundary conditions describe all processes in electrodynamics.

There is no general solution of the system of Maxwell equations with boundary conditions similar to the Lienard-Viechert solution for the fields produced by charged particles moving along some trajectories at a given low in free space. It means that private problems must be solved separately for every concrete case. At that when the boundary conditions exist, interactions of particles with surrounding media and intrabeam interactions of particles are essential then the beam density and beam current can not be given and the dynamical Lorentz equations must be added. Below we will consider the case when the beam density and the density of the beam current (particle trajectories) are given.

One of the possible simplifications of the solution of the Maxwell equations is to transform linear Maxwell equations to the equations of the second order relative to the field strengths or potentials.

First of all the Maxwell equations can be transformed to the equations separately for the electric and magnetic fields. For this purpose we can differentiate equation (1.b) with respect to t, use equation (1.c) and employ the vector identity $rot\, rot\vec{F} = grad\, div\, \vec{F} - \Delta\vec{F}$, where Δ is the Laplacian operator. Such a way we will receive the equation for the electric field strength and then by analogy we will receive the equation for the magnetic field strength. They are

$$\Box\vec{E} = \frac{4\pi}{c^2}\dot{\vec{J}} + 4\pi\, grad\, \rho, \quad (a) \qquad \Box\vec{H} = -\frac{4\pi}{c} rot\, \vec{J}. \quad (b) \tag{16}$$

where $\Box = \Delta - \partial^2/c^2\partial t^2$ is the d'Alembertian operator, $\dot{\vec{J}} = \partial\vec{J}/\partial t$.

The equations (16) are the nonhomogeneous linear equations of the second order. We must add the equations (1.a), (1.d) to the system of the equations (16). It means that we have again a system of two vector and two scalar equations (in components they are eight equations) for six unknown components of the electric and magnetic field strengths E_i, H_i.

The divergence of the equation (16.a) leads to a more general continuity equation $(\partial/\partial t)(\partial\rho/\partial t + div\vec{J}) = 0$ which is valid when the continuity equation $\partial\rho/\partial t + div\vec{J} = 0$ is valid.

The solution of the Maxwell equations will be the solution of these second order equations. The second order equations are another equations. Strictly speaking they are not equivalent to Maxwell equations. We must check theirs solutions by substituting these solutions into the linear Maxwell equations to reject unnecessary solutions. This is very difficult problem even for simple cases. A way out can be found by introducing of electromagnetic field potentials. The vector potential $\vec{A}$ and scalar potential φ are introduced by the equations $\vec{H} = rot\vec{A}$, $\vec{E} = -grad\varphi - (1/c)(\partial\vec{A}/\partial t)$. In this case both from Maxwell equations and from the equations (16) it follows the equations for vector and scalar potentials

$$\Box\vec{A} = -\frac{4\pi}{c}\vec{J} \quad (a), \qquad \Box\varphi = -4\pi\rho \quad (b) \tag{17}$$

and additional condition coupling the potentials (Lorentz gauge)

$$div\vec{A} = -\frac{1}{c}\frac{\partial\varphi}{\partial t}. \tag{18}$$

It is convenient to use the electric and magnetic Hertz vectors as well. They permit to simplify the solutions of the problem of propagation of waves in resonators and free space which is described by the homogeneous wave equations ($\rho = 0$, $\vec{J} = 0$). Both the electric and magnetic Hertz vectors $\vec{\Pi}^e$, $\vec{\Pi}^m$ are introduced by the same expressions

$$\vec{A} = \frac{1}{c}\frac{\partial\vec{\Pi}^{e/m}}{\partial t}; \qquad \varphi = -div\vec{\Pi}^{e/m}. \tag{19}$$

Such way defined potentials $\vec{A}$ and φ will satisfy the equation (11) simultaneously.

Different superscripts e/m in this case are used on the stage of introduction of the connection between electric and magnetic field strengths through Hertz vectors. The electric field strength can be expressed through the electric and magnetic Hertz vector by the equations

$$\vec{E} = grad\, div\vec{\Pi}^e - \frac{1}{c^2}\frac{\partial^2\vec{\Pi}^e}{\partial t^2}; \qquad \vec{H} = \frac{1}{c}\frac{\partial}{\partial t}rot\vec{\Pi}^e, \tag{20}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}rot\vec{\Pi}^m, \qquad \vec{H} = grad\, div\vec{\Pi}^m - \frac{1}{c^2}\frac{\partial^2\vec{\Pi}^m}{\partial t^2}. \tag{21}$$

These manipulations are valid because of both definitions (20) and (21) satisfy Maxwell equations (1) and equations (16). This is because of homogeneous wave equations for electromagnetic fields

$$\Box\vec{F} = 0 \tag{22}$$

are symmetric relative to fields $\vec{E}$, $\vec{H}$ ($\vec{F} = \vec{E}, \vec{H}$). If $\vec{E}$ and $\vec{H}$ are some solutions of the homogeneous Maxwell equations (1b), (1c) then vectors $\vec{E}' = -\vec{H}$ and $\vec{H}' = \vec{E}$ will satisfy these and another Maxwell equations as well.

In general case the problem may be reduced to solving of wave equation if potentials $\vec{\Pi}^e$, $\vec{\Pi}^m$ will be introduced simultaneously in the form [10]

$$\vec{E} = grad\, div\Pi^e - \frac{1}{c^2}\frac{\partial^2\vec{\Pi}^e}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}rot\vec{\Pi}^m, \qquad \vec{H} = \frac{1}{c}\frac{\partial}{\partial t}rot\vec{\Pi}^e + grad\, div\vec{\Pi}^m - \frac{1}{c^2}\frac{\partial^2\vec{\Pi}^m}{\partial t^2}. \tag{23}$$

We can be convinced that $\vec{E}$ and $\vec{H}$ described by (23) fulfil to Maxwell equations at $\rho = \vec{J} = 0$ when vectors $\vec{\Pi}^e$ and $\vec{\Pi}^m$ fulfil the wave equation (22) with replaced $\vec{F} \rightarrow$ on $\vec{\Pi}^e$ and $\vec{\Pi}^m$.

Equation (22) is valid for each component of vectors $\vec{\Pi}^e$ and $\vec{\Pi}^m$. That is why it is possible to use scalar wave equation

$$\Box U = 0 \tag{24}$$

and identify its solution U with one of components of vectors $\vec{\Pi}^e$ or $\vec{\Pi}^m$ and the rest components of these vectors equate to zero (say we can take $\vec{\Pi}^e = \vec{e}_x \cdot 0 + \vec{e}_y \cdot 0 + \vec{e}_z \cdot U$, $\vec{\Pi}^m = 0$). Substituting the constructed such a way vector with one component in (16) we will find the electromagnetic field strengths $\vec{E}$, $\vec{H}$ which satisfy the Maxwell and wave equations. Then we can identify the same solution with another component of the Hertz vector, equate the rest components to zero and calculate another electromagnetic field strengths $\vec{E}$, $\vec{H}$ which satisfy the Maxwell and wave equations as well. After we will go through all compositions with components then we will have a set of six different solutions for field strengths $\vec{E}$, $\vec{H}$. These solutions will be six electromagnetic waves with different structures. Sum of these solutions with some coefficients will be a solution of the Maxwell equations as well. This will be algorithm of electromagnetic field determination through Hertz vector.

Equation (24) has many different solutions. We must find such solutions which will correspond to the problem under consideration to a considerable extent. Below we will deal with monochromatic light beams of the limited diameter related with resonator modes. In general case such beams can be written in the form

$$U(x,y,z) = V(x,y,z)e^{i(kz-\omega t)} \tag{25}$$

where $V(x,y,z)$ is a function of coordinate slowly varying in comparison with $\exp i(kz-\omega t)$. A complex form of values will be used for computations and then we will proceed to a real part of the form.

Substituting (25) into (24) and taking into account the slow variation of $V(x,y,z)$ compared with $\exp i(kz-\omega t)$ that is the condition $|\partial^2 V/\partial z^2| \ll 2k|\partial V/\partial z|$ and the condition $k = \omega/c$ we will receive the equation

$$i\frac{\partial V}{\partial z} + \frac{1}{2k}(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}) = 0 \tag{26}$$

which describes a space limited beam.

In the general case the limited in the transverse direction wave propagating in free space or in a resonator have rather complicated structure. That is why it is desirable to find full, orthogonal set of fundamental waves with the well known feature of propagation. Then an arbitrary wave may be expanded into series of these waves. Different series of fundamental waves can be found for this problem and the arbitrary wave can be expanded into one or another series. The method of separation of variables is used to solve the wave equation. For example, in the Cartesian coordinates $V(x,y,z) = X(x,y,z)\cdot Y(x,y,z)$ and in the cylindrical coordinates $V(x,y,z) = G(u)\Phi(\varphi)\exp[ikr^2/2q(z)]\cdot\exp[iS(z)]$, where r and φ are cylindrical coordinates on a plane transverse to z, $u = r/w(z)$. These solutions are considered in [10].

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